Eighth Annual Upper Peninsula
High School Math Challenge

Northern Michigan University (Marquette, MI, USA)
Saturday 8 April 2017

Individual Problems — Solutions

PROBLEM 1

The sum of three numbers is 98. The first number is 2/3 of the second, and the second is 5/8 of the third. What is the second number?

Answer: 30

$$x + y + z = 98, \quad x = \frac{2y}{3}, \quad y = \frac{5z}{8} \Rightarrow z = \frac{8y}{5}$$

$$\therefore \frac{2y}{3} + y + \frac{8y}{5} = 98$$

$$y \cdot \left(\frac{2}{3} + 1 + \frac{8}{5} \right) = y \cdot \frac{10 + 15 + 24}{15} = y \cdot \frac{49}{15} = 98$$

$$\frac{49y}{15} = 98 \Rightarrow y = \frac{98 \cdot 15}{49} = 30$$

Also, $y = 30 \Rightarrow x = 20$ and $z = 48$

Alternate solution.:

Let x = the third number.

Then the second number = $\dfrac{5x}{8}$

first number = $\left(\dfrac{2}{3} \right)\left(\dfrac{5x}{8} \right) = \dfrac{5x}{12}$

$$\frac{5x}{8} + \frac{5x}{12} + x = \left(\frac{15 + 10 + 24}{24} \right) \cdot x = \frac{49x}{24} = 98,$$

and $x = 48$. $\therefore \dfrac{5x}{8} = \dfrac{5 \cdot 48}{8} = 30$
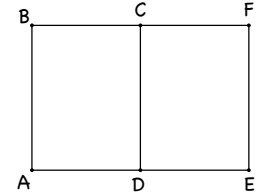
PROBLEM 2

Ms. Daniels recorded the test scores for her algebra class of 25 students. The class average on the test was 72. Unfortunately, Samantha's test score of 86 was incorrectly recorded as 36. What was the correct test average?
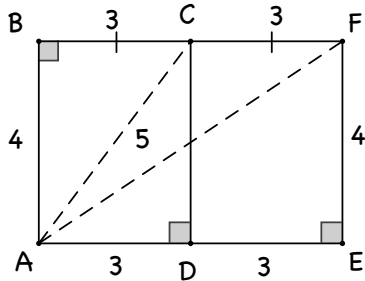
Answer: 74

$$25 \times 72 = 1800$$
$$(25 \times 72) + 50 = 1850$$
$$1850 \div 25 = 74$$

PROBLEM 3

ABCD and DCFE are coplanar rectangles that share a common side, CD.
If AB = 4 cm, AC = 5 cm, and BC = CF, what is the length of AF?

Answer: $2\sqrt{13}$ **cm**

AB = 4, AC = 5 $\Rightarrow$ BC = 3 (right $\triangle$)
BC = CF = 3
$\therefore$ AE = BF = 6, AB = EF = 4 (rectangles)
AF = $\sqrt{36 + 16}$ = $\sqrt{52}$ = $2\sqrt{13}$

PROBLEM 4

One line has a slope of m and a y-intercept of 2. A different line has a slope of 2 and a
y-intercept of m. At what coordinates, in terms of m, must the lines intersect?

Answer: (1, m+2)

The equations for the given lines are
    y = mx + 2 and y = 2x + m,
so they intersect when mx + 2 = 2x + m.

$\therefore$ mx − 2x = m − 2,
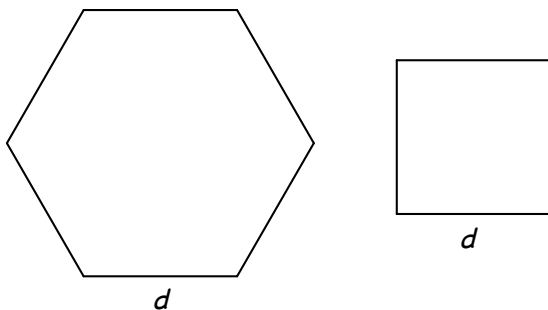 x·(m − 2) = m − 2 $\Rightarrow$ x = 1 (provided m ≠ 2).

 y = mx + 2 = m·1 + 2 = m + 2.

Note: m ≠ 2 because if m = 2 the lines are not distinct.

PROBLEM 5

A regular hexagon and a square each have a side of length d inches. The area of the hexagon is $384\sqrt{3}$ square inches. What is the area of the square?
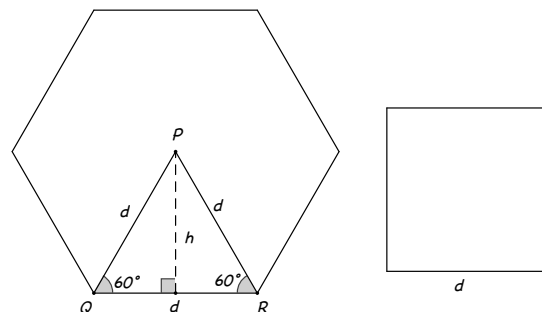

d

d

Answer: 256 in^2

$h = \dfrac{d\sqrt{3}}{2}$ area $\triangle PQR = \dfrac{d}{2}\cdot\dfrac{d\sqrt{3}}{2}$

area hexagon $= 6\cdot\dfrac{d^2\sqrt{3}}{4} = 384\sqrt{3}$

$d^2 = 4\cdot\dfrac{384}{6} = 4\cdot64 = 256$ square inches $=$ area square

PROBLEM 6

The sum of seven consecutive integers is 980. Which of the integers is/are prime?

Answer: 137 and 139

Sum of 7 integers = 980
Average = 140 (center term)
Integers are: 137 + 138 + 139 + 140 + 141 + 142 + 143
Of these seven, the primes are 137 and 139
  (2 divides the evens; 3 divides 141; 11 divides 143)

OR

$x + (x + 1) + (x + 2) + \ldots + (x + 6) = 980$
$7x + 21 = 980$
$7x = 959$
$x = 137$ and the integers are 137 through 143 (as above)

PROBLEM 7

A certain game uses unusual dice that have their six faces labeled 1, 2, 3, 5, 7, 9. If two of these dice are rolled and the numbers on the upper faces are added, what is the probability of rolling a sum of 10?

<u>Answer: P (sum of 10)</u> = $\dfrac{5}{36}$

	1	2	3	5	7	9
1	2	3	4	6	8	**10**
2	3	4	5	7	9	11
3	4	5	6	8	**10**	12
5	6	7	8	**10**	12	14
7	8	9	**10**	12	14	16
9	**10**	11	12	14	16	18

PROBLEM 8

Four congruent rectangles are arranged as shown to form a large rectangle. The area of the large rectangle is 768 cm². What is the area of a square that has the same perimeter as the large rectangle?
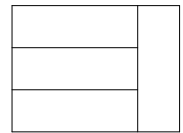
<u>Answer: 784 cm²</u>

Let x = shorter side and y = longer side.

Area of small rectangle = $x \cdot y = \dfrac{768}{4} = 192$

From the diagram, $3x = y$, $\therefore$ $x \cdot 3x = 3x^2 = 192$
  $x^2 = 64$ $\Rightarrow$ $x = 8$, $y = 24$
Perimeter large rectangle = $2 \cdot (24 + 32) = 112$ cm

$P = 112 \Rightarrow$ side of desired square = $s = \dfrac{112}{4} = 28$

area of square = $28^2 = 784$ cm²

PROBLEM 9

A neon billboard flashes a blue sign for Product A every 39 seconds, and it flashes a red sign for Product B every 20 seconds. If both signs flash at the same time at 8:00 pm, when is the next time they flash together?
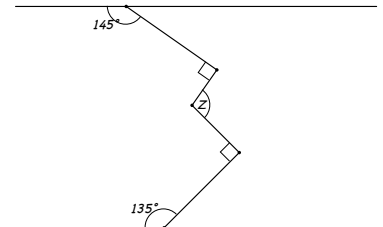
<u>Answer: 8:13 pm</u>

blue flash (b): 39 seconds
red flash (r): 20 seconds
$39b = 20r$
$[39, 20] = 1$ (relatively prime)
$\therefore$ LCM $(39, 20) = 780$ seconds $(39 \times 20 = 780)$
780 seconds = 13 minutes
simultaneous flash every 13 minutes
next coincidence at 8:13 pm

PROBLEM 10

Find the measure of the angle z in the diagram, given that the two horizontal segments are parallel.

<u>Answer: 100°</u>

Construct lines through B, C, D parallel to the given segments.
$\angle FAB = 145°$ given
$\angle ABG = 35°$ supplement to $\angle FAB$
$\angle GBC = 55°$ complement to $\angle ABG$
$\angle BCJ = 55°$ altetnate interior angles
$\angle IED = 135°$ given
$\angle EDH = 45°$ supplement to $\angle IED$
$\angle HDC = 55°$ complement to $\angle EDH$
$\angle DCJ = 45°$ altetnate interior angles

$\angle Z = \angle BCD = \angle BCJ + \angle DCJ = 55° + 45° = 100°$